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ABSTRACT

Several examples are given of a variety of nonstandard problems, not ordinarily found in texts or courses, now accessible to students who have programmable calculators. These include: (1) finding limits; (2) evaluating infinite series; (3) calculating finite series; (4) computing variable length products; (5) solving equations; (6) searching for data; and (7) developing proofs. These types of problems are discussed and reasons are given for including them in the undergraduate curriculum and solving them with calculators. (MP)

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Because of Programmable Calculators, Why Avoid These Problems Any Longer?

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Now that hand held programmable calculators are inexpensive and widely available they can have a major impact on the teaching of college mathematics. For example, it is now possible for the typical undergraduate student to solve a variety of nonstandard problems not ordinarily found in texts or courses.

The purpose of this note is to give several examples of the types of problems now accessible to students who have programmable calculators. The selection of problems is not intended to be exhaustive but it is designed to indicate the variety of problems which are now solvable at the undergraduate level.

Finding Limits

In most calculus texts limit problems are solved by referring students back to several fundamental limits:

$$\lim_{x\to\infty} 1/x = 0 \tag{1}$$

$$\lim_{n\to\infty} x^n = \begin{cases} \infty & |x|>1 \\ 1 & x=1 \\ 0 & |x|<1 \end{cases} \tag{2}$$

$$\lim_{n\to\infty} \sqrt[n]{n} = 1 \tag{3}$$

$$\lim_{x\to 0} (\sin x)/x = 1 \tag{4}$$

$$\lim (1 + (1/n))^n = e \tag{5}$$

While these limits are sometimes proved in class, they are often taken for granted. Most other limits are then calculated by algebraically manipulating to refer back to the fundamental limits, for example,

$$\lim_{x\to 0} (\sin^2 x)/x^3 = (\lim_{x\to 0}((\sin x)/x)^2)(\lim_{x\to 0} 1/x) = 1^2\cdot 0 = 0 \tag{6}$$

Although calculators can not prove what limits are, they can establish a quick and very strong indication of the numerical value of any specific limit. In particular, each of the fundamental limits can be "verified" directly using calculators with the advantage that students "see" a value happening as a limit of a sequence rather than the outcome of an algebraic procedure. Moreover, with

the calculator, almost any limit can be discovered or verified without the use of advanced techniques. For example, while

$$\lim_{x\to\infty} x^e/e^x \tag{7}$$

can be evaluated by three repetitions of L'Hospital's rule, the limit can be found easily on the calculator without knowing any advanced techniques or even any other limits.

EVALUATING INFINITE SERIES

In second or third term calculus courses, infinite series play an important role but rarely are students asked to calculate actual values for series and even then only for geometric series such as

$$\sum_{n=1}^{\infty} 1/3^n = 1/3 + 1/9 + 1/27 + \ldots + 1/3^n + \ldots \tag{8}$$

or telescoping series such as

$$\sum_{n=1}^{\infty} 1/(n(n+1)) = 1/(1\cdot 2) + 1/(2\cdot 3) + \ldots + 1/(n(n+1)) + \ldots \tag{9}$$

Evaluating other types of series has been omitted in the past because exact calculations require techniques that are too advanced and approximations are too tedious.

Programmable calculators surmount these limitations. Consider for example the series

$$\sum_{n=1}^{\infty} n^2/2^n = 1/2 + 4/4 + 9/8 + 16/16 + \ldots + n^2/2^n + \ldots \tag{10}$$

and the series

$$1/2 + 1/3 + 1/7 + 1/43 + 1/1807 + \ldots + \ldots \tag{11}$$

where successive denominators are defined by recursion as $d_i = d_i^2 - d_{i-1} + 1$ and $d_1 = 2$.

Because of the recursive and looping capabilities of programmable calculators, the calculation of the first n terms of these series becomes a simple programming exercise. In fact, the programmable calculator can evaluate any infinite series from the fundamental definition that a series is the limit of its partial

sums.

CALCULATING FINITE SERIES

In both precalculus and calculus courses finite series play a role. At the precalculus level students encounter finite series when induction is taught. For example, students are asked to prove by induction

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)}{4} \qquad (12)$$

At the calculus level finite series occur as approximations of an area under a curve and serve to motivate the definition of an integral. For example, the area between $x = 0$ and $x = 1$ under the curve $y = \sqrt{x}$ can be approximated with

$$\text{Area} = \sum_{k=1}^{n} \frac{1}{2}\left(\sqrt{\frac{2k-1}{2n}}\right) = \frac{1}{n}\left(\sqrt{\frac{1}{2n}}\right) + \frac{1}{n}\left(\sqrt{\frac{3}{2n}}\right) + \ldots + \frac{1}{n}\left(\sqrt{\frac{2n-1}{2n}}\right) \qquad (13)$$

Rarely are students asked to actually calculate either of these types of finite sums for specific values of n.

Programmable calculators perform calculations of finite sums rather quickly, even for relatively large values of n, e.g., n = 30. Like programming infinite series, a finite series is programmed as a running total of individually calculated terms. In order to stop the addition and obtain the total of the first n terms, a student can either decide by eye whether n terms have been totaled or program the calculator to actually make that decision.

With programmable calculators, students now have at their disposal apparatus that allows them to solve the following problems.

What is the smallest value of n for which $\sum_{k=1}^{n} 1/2k > 2$? (14)

Approximate $\int_0^3 e^{-x^2}\,dx$ using rectangles of width .1 each. (15)

Induction on the following formula begins properly but (16) the induction step itself seems to fail. Test the formula

to see whether it is perhaps false, using n = 10.

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = 2.5n^2 - 3.5n + 2$$

Show that the sum of the first 10 Fibonacci numbers is 11 times the seventh Fibonacci number, i.e., $\sum_{k=1}^{10} f(k) = 11 \cdot f(7)$ where $f(1)=1$, $f(2)=2$, and $f(n)=f(n-1) + f(n-2)\ \forall\ n > 2$. (17)

Programming finite series and infinite series are alike in that the looping capability of the calculator computes the running total as a sum of individually calculated terms. For finite series, the calculators decision making capability permits the looping to halt, thereby producing a finite sum.

COMPUTING VARIABLE LENGTH PRODUCTS

Students in calculus and elementary probability are exposed to relatively few variable length product formulas like factorials

$$n! = n(n-1)(n-2)\ldots(3)(2)(1), \tag{18}$$

the binomial coefficients

$$\binom{n}{r} = (n(n-1)\ldots(n-r+1))/(r(r-1)\ldots(1)), \tag{19}$$

and the probability of obtaining r heads on one toss of n (not necessarily fair) coins, each with probability p of falling heads up

$$P(n,r) = \binom{n}{r}.p^{r}.(1-p)^{n-r} \tag{20}$$

Such quantities are rarely computed, except for small values of n or for values appearing in prepared tables because the calculations are so cumbersome.

As an example of other problems of this type that students often encounter but rarely investigate, consider the well known birthday problem.

Define the probability that at least two persons in a room of n people share the same birth date to (21)

be P(n) where P(n)= 1 - ((365/365)(364/365)...(366-n)/365)).

The birthday problem requires finding the smallest value of n for which P(n) exceeds 50% and is easily done by experimentation once a program for the calculation has been written.

Programming a calculator to evaluate a variable length product is similar to programming the calculation of a finite series. A finite series can be calculated as a running sum of (finitely) many terms; analogously, a variable length product can be calculated as a running product of (finitely) many factors.

SOLVING EQUATIONS

While undergraduate mathematics students are expected to solve equations such as

$$x^4 - 5x^2 + 4 = 0 \qquad (22)$$

$$\sin x = \cos x \qquad (23)$$

$$2^x = 23 \qquad (24)$$

they are not expected to solve these similar kinds of problems

$$x^4 - 5x^3 + 4 = 0 \qquad (25)$$

$$\sin x = x^2 \qquad (26)$$

$$2^x = 23x \qquad (27)$$

nor more advanced problems such as

Find the value of $\sqrt{10-\sqrt{10+\sqrt{10-\sqrt{10+\ldots}}}}$ (28)

Where does the curve y = tanhx cross the curve y = tan x? (29)

What value(s) of x maximize the function $(e^x - \ln x)$? (30)

What is the smallest value x_0 for which all $x > x_0$ satisfy $\ln x < x-4$? (31)

However, with general iterative solution techniques each

of these problems can be readily solved on the programmable calculator. For a comprehensive treatment of this topic including calculator programs see [3].

SEARCHING FOR DATA

A great deal of mathematical activity involves the collection and analysis of data. Too often the undergraduate does not participate in data collection and analysis experiences because of the amount of time required. The programmable calculator is a valuable tool in this context because it makes it easier for students to explore data collection problems. Here are two examples of problems in which data collection and analysis are facilitated by the use of programmable calculators.

Find all the ways that a rope joined at both ends and with 360 equally spaced knots can be stretched taut into a right triangle with knots at the vertices. (32)

This problem can be done by finding all the integer solutions to the system

$$a^2 + b^2 = c^2$$

$$a + b + c = 360$$

$$a + b > c$$

where a, b, and c are integers.

This problem illustrates the use of programmable calculator to search for data satisfying necessary restrictions.

Decide how faces of a cube should be painted red if the probability of the cube landing with a red face up exactly two out of three times is to be maximized. (33)

In this problem the programmable calculator can be used to calculate each of the probabilities for the various different ways the cube can be painted and at the same time locate which one corresponds to the greatest probability. The programmable

calculator's decision making capability allows it to locate a maximum of a finite sequence.

DEVELOPING PROOFS

There are at least two ways in which programmable calculators can be used in developing proofs. They can generate data from which conjectures may be made, leading to a need for a mathematical proof, perhaps one by induction. In addition to just generating data and stimulating conjectures, calculator results can occasionally be used as an essential part of a proof. As examples of each of these ways consider the following two problems.

(34) Given the Fibonacci sequence

$1, 2, 3, 5, 8, 13, \ldots, S_n, \ldots$

where $S_n = S_{n-1} + S_{n-2}$, generate and examine two sequences of data as follows:

i. the sum of the squares of the first n Fibonacci terms for various values of n

ii. the products of adjacent Fibonacci numbers.

What general relationship exists between the corresponding sequence terms?

The data collection and analysis yields

$$S_1^2 + S_2^2 + S_3^2 + \ldots + S_n^2 = (S_{n+1})(S_n) + 1$$

which can be proved by induction.

(35) Notice that $36 = (3)(2) + (6)(5)$. Find all integers that equal the sum of the products of each digit with one less than that digit, that is

$$n = \sum d(d-1)$$

where the summation is over all digits, d, of n.

Searching the first one thousand natural numbers yields three results, 2, 36 and 86. It seems reasonable to conjecture that there are no more numbers with this property. To explore

this conjecture notice that the digit d cannot exceed 9 so $\sum d(d-1)$ cannot exceed (9)(8)(number of digits of n). Based on this fact it is possible to prove analytically that every $n \geq 1000$ has its $d(d-1)$ being less than n itsel (and hence never equal to n). Importantly, this analytic proof is valid only for $n \geq 1000$ since some numbers less than 1000 have their $\sum d(d-1) > n$, while others have $n = \sum d(d-1)$ as already indicated. However, the analytic proof for $n \geq 1000$ together with the calculator search for $1 \leq n < 1000$ constitutes a proof valid for all natural numbers that only for n = 2, 36, or 86 does $n = \sum d(d-1)$.

Thus, as illustrated by this problem, calculator searches can in certain situations become essential pieces of a proof. For another treatment of this idea including several number theory problems solvable by some combination of calculator (computer) calculation and mathematical analysis, see [1] .

CONCLUSION

As illustrated by the examples presented in this article there is a wide variety of nonstandard problems rarely found in text books, which are accessible to students with programmable calculators. Because the cost is low enough that it is not unreasonable to require each student to own a programmable calculator, thes problems can be included in the undergraduate curriculum. Moreover, we authors believe that these problems should be included and solved with calculators for the following reasons. Programmable calculators

a. expand the level of difficulty of problems that can be solved;

b. provide general techniques for evaluating limits, series and variable length products as well as for solving equations;

c. force a review of the definitions of limits and series when these are calculated;

d. motivate definitions of integrals and proofs by induction;

e. permit exploration with permutation, combination and other variable length product formulas;

f. allow verification of frequently stated but otherwise unjustified facts (like the result of the birthday problem);

g. facilitate the gathering of data which is often useful in making conjectures and occasionally useful in completing proofs;

h. search for a maximum or a minimum of a finite sequence; and

i. locate all data values satisfying g ven restrictions.

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